PERGAMON

International Journal of Solids and Structures 37 (2000) 4353–4375

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

An approximate solution for thin rectangular orthotropic/isotropic strips under tension by line loads

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Received 23 July 1998

Abstract

An approximate analytical solution for stresses and displacements in thin rectangular orthotropic or isotropic strips subjected to tension by longitudinal line loads is presented. It is assumed that changes in the transverse direction of the strip middle surface are small, in comparison to the changes in the longitudinal direction, and can be ignored in the shear strain–displacement relation. The solution is given for linearly and uniformly distributed line loads in the longitudinal direction, generally (symmetrically) distributed in the transverse direction. Some simple examples are analyzed, and compared to exact solutions of the plane theory of elasticity and the finite element analysis. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Thin rectangular strips; Orthotropic/isotropic materials; Tension; Line loads

1. Introduction

The problem of the stress estimation in thin rectangular strips of isotropic or orthotropic materials subjected to tension by line loads can be considered, in general, by methods of the plane theory of elasticity (Papkovich, 1939; Kurdyumov et al., 1963). Meanwhile, some simple solutions can be obtained for a number of simple cases, only. A typical example is an isotropic strip loaded by line loads along its longitudinal edges presented by the Fourier series of cosine or sine mode shapes. The solution by cosine load modes assume that the longitudinal normal stress and transverse displacement vanish at the strip ends (Filon, 1903; Shade, 1951; Abdel-Sayed, 1969), while sine load modes assume that the shear stress and longitudinal displacement vanish at the strip ends (Ribière, 1898; von Karman, 1924; Beschkin,

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PII: S0020-7683(99)00152-3

1937–38; Fukuda, 1963). The first solution corresponds to the flanges of an antisymmetrically loaded multispan thin-walled girder, while the second one corresponds to the symmetrically loaded multispan girder. For the orthotropic strip the solution would be only more complicated due to the definition of the state of stress for orthotropic materials (Kurdyumov et al., 1963; Abdel-Sayed, 1969; Senjanović and Fan, 1992).

Some simple approximate solutions based on assumptions about stresses and strains for isotropic strips can be found in the literature. There, some acceptable results for the maximum longitudinal stress are obtained; if the length of the strip is not too small compared to the breadth of the strip (Boytzov and Paliy, 1979; Pavazza and Plazibat, 1997). Such solutions can be very suitable due to their simplicity, especially in the early design stage of structures (in civil and marine engineering, etc.).

In this paper, a simple approximate solution is investigated for the thin rectangular orthotropic strips. A general disposition of the longitudinal loads, within the longitudinal edges of the strip, is assumed. Loading along the longitudinal edges, and along the central longitudinal section will be only special cases; as well as the solution for isotropic strips. The results are proved using exact solutions of the plane theory of elasticity and the finite element method.

2. Basic relation

A thin rectangular strip of length l and breadth b, thickness t, is loaded symmetrically, with respect to the central longitudinal axes x, along longitudinal sections, at $y = \pm c$, by line loads $T = T(x)$, in the strip middle surface (Fig. 1).

A state of plane stress is assumed, where for orthotropic materials:

$$\frac{E_x}{E_y} = \frac{\nu_x}{\nu_y}, \tag{1}$$

where E_x and E_y are the moduli of elasticity, in the longitudinal and transverse direction, respectively; ν_x and ν_y are the Poisson's ratios, with respect to tension in longitudinal and transverse direction, respectively;

$$\epsilon_x = \frac{1}{E_x}(\sigma_x - \nu_x\sigma_y), \quad \epsilon_y = \frac{1}{E_y}(\sigma_y - \nu_y\sigma_x), \quad \gamma_{xy} = \frac{\tau_{xy}}{G}, \tag{2}$$

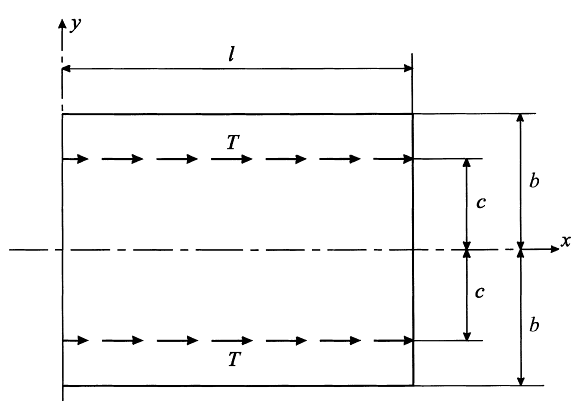

Fig. 1. Thin rectangular strip subjected to tension by line loads.

where $\epsilon_x = \epsilon_x(x, y)$ and $\epsilon_y = \epsilon_y(x, y)$ are strains in the longitudinal and transverse direction, respectively; $\sigma_x = \sigma_x(x, y)$ and $\sigma_y = \sigma_y(x, y)$ are the normal stresses in the longitudinal and transverse direction, respectively; $\gamma_{xy} = \gamma_{yx} = \gamma_{xy}(x, y)$ is the shear strain; $\tau_{xy} = \tau_{yx} = \tau_{xy}(x, y)$ is the shear stress; G is the shear modulus.

It will be assumed that changes of middle surface in the transverse direction are small compared to the changes in the longitudinal direction, and can be ignored in the shear strain–displacement relation.

The strain–displacement relations may then be written as follows

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y}, \tag{3}$$

where $u = u(x, y)$ and $v = v(x, y)$ are the displacements, in the longitudinal and transverse direction, respectively. Consequently, the compatibility condition takes the following simple form

$$\frac{\partial \epsilon_x}{\partial y} = \frac{\partial \gamma_{xy}}{\partial x}. \tag{4}$$

Referring to (2), the following equation may then be obtained

$$\frac{\partial \sigma_x}{\partial y} - \nu_x \frac{\partial \sigma_y}{\partial y} = \frac{E_x}{G} \times \frac{\partial \tau_{xy}}{\partial x}. \tag{5}$$

The equilibrium conditions may be obtained for a portion of the strip (Fig. 2), as follows

$$\int_{y^*} \frac{\partial \sigma_x}{\partial x} \mathrm{d}y + \frac{q_x^*}{t} - \tau_{yx} = 0; \quad \int_{y^*} \frac{\partial \tau_{xy}}{\partial x} \mathrm{d}y - \sigma_y = 0, \tag{6}$$

where

$$q_x^* = q_x^*(x, y) = \begin{cases} T \mid -T(y-c)^0 & (0 \le y \le b), \\ -T \mid +T(y+c)^0 & (0 \ge y \ge -b), \end{cases} \tag{7}$$

where $y^* = b - y$ $(-y^* = b + y)$ is the coordinate from the longitudinal edge, where $\sigma_y = 0$; the vertical lines denotes (Clebshe): for $y \le c$ $(y \ge -c)$ the part left of the line, for $y \ge c$ $(y \le -c)$ the line must be deleted. The total line load is

$$q_x = 2q_x^*(x, b) = 2T. \tag{8}$$

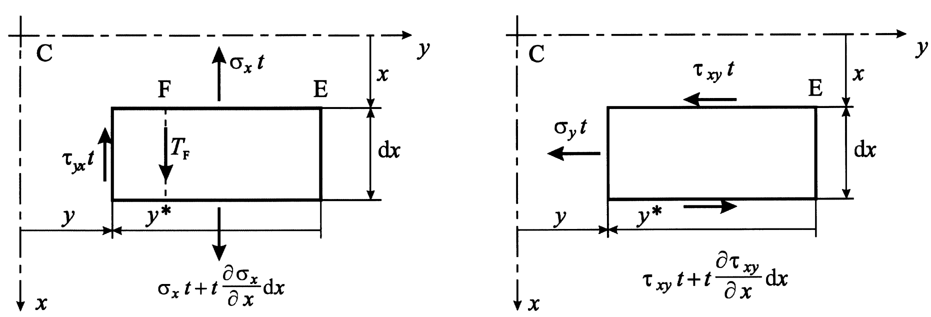

Fig. 2. Equilibrium of a portion of the strip.

Referring to (6), one may write

$$\tau_{yx} = \tau_{xy} = \int_{y^*} \frac{\partial \sigma_x}{\partial x} \mathrm{d}y + J^* \frac{q_x}{t}, \tag{9}$$

where

$$J^* = J^*(y) = \frac{q_x^*}{q_x} = \begin{cases} \frac{1}{2} \mid -\frac{1}{2}(y-c)^0 & (0 \le y \le b), \\ -\frac{1}{2} \mid +\frac{1}{2}(y+c)^0 & (0 \ge y \ge -b). \end{cases} \tag{10}$$

The normal force $N=N(x)$ can be defined as follows

$$\int_A \sigma_x \, \mathrm{d}A = N, \tag{11}$$

where A is the strip cross-section area: $A = 2bt$. Then, from the equilibrium

$$\frac{\mathrm{d}N}{\mathrm{d}x} = -q_x. \tag{12}$$

According to (3) and (2), the displacement in the longitudinal direction can be expressed as follows

$$u = u_C + \frac{1}{G} \int_0^y \tau_{xy} \, \mathrm{d}y, \tag{13}$$

where $u_C=u_C(x)$ is an integration constant, that is the displacement of the cross-section centroid (the origin of the y coordinate). Referring to (3), the longitudinal strain can then be written as

$$\epsilon_x = \frac{\mathrm{d}u_C}{\mathrm{d}x} + \frac{1}{G} \int_0^y \frac{\partial \tau_{xy}}{\partial x} \mathrm{d}y. \tag{14}$$

Referring to (2) and (14), the longitudinal normal stress may be expressed as

$$\sigma_x = E_x \frac{\mathrm{d}u_C}{\mathrm{d}x} + \frac{E_x}{G} \int_0^y \frac{\partial \tau_{xy}}{\partial x} \mathrm{d}y + \nu_x \sigma_y. \tag{15}$$

The transverse normal stress, given by (6), may be written as

$$\sigma_y = \sigma_{y0} - \int_0^y \frac{\partial \tau_{xy}}{\partial x} \mathrm{d}y, \quad \sigma_{y0} = \int_b \frac{\partial \tau_{xy}}{\partial x} \mathrm{d}y. \tag{16}$$

The longitudinal normal stress may then be rewritten as follows

$$\sigma_x = E_x \frac{\mathrm{d}u_C}{\mathrm{d}x} + \nu_x \sigma_{y0} + \frac{E_x}{G} \int_0^y \frac{\partial \tau_{xy}}{\partial x} \mathrm{d}y. \tag{17}$$

Referring to (3) and (2), the transverse displacement is defined by

$$v = \frac{1}{E_y} \int_0^y (\sigma_y - \nu_y \sigma_x) \mathrm{d}y, \tag{18}$$

or by employing (16) and (17)

$$v = \frac{1}{E_y}\left\{\left[(1-\nu_x\nu_y)\sigma_{y0} - \nu_y E_x \frac{\mathrm{d}u_C}{\mathrm{d}x}\right]y - \left[1+\nu_y\left(\frac{E_x}{G}-\nu_x\right)\right]\int_0^y\int_0^y \frac{\partial \tau_{xy}}{\partial x}\mathrm{d}y\,\mathrm{d}y\right\}. \tag{19}$$

3. Uniformly distributed loads

If the shear stress is given as

$$\tau_{xy} = \tau_{xy}(y), \tag{20}$$

then taking into account (16)

$$\sigma_y = 0. \tag{21}$$

From (17) one obtains

$$\sigma_x = E_x \frac{\mathrm{d}u_C}{\mathrm{d}x} \tag{22}$$

and from (11)

$$E_x A \frac{\mathrm{d}u_C}{\mathrm{d}x} = N. \tag{23}$$

Thus,

$$\sigma_x = \frac{N}{A}. \tag{24}$$

Taking into account (12), the following equation may be written

$$E_x A \frac{\mathrm{d}^2 u_C}{\mathrm{d}x^2} = -q_x. \tag{25}$$

By substituting (24) into (9), taking into account (12), the shear stress may be expressed as follows

$$\tau_{xy} = \left(J^* - \frac{A^*}{A}\right)\frac{q_x}{t}, \tag{26}$$

where

$$A^* = \begin{cases} (b-y)t & (0 \le y \le b), \\ -(b+y)t & (0 \ge y \ge -b). \end{cases} \tag{27}$$

It follows, according to (18), that

$$q_x = \text{const.} \tag{28}$$

From (8), it follows that the line loads must be distributed uniformly (T = const).

The solution given by (21), (24) and (26) satisfies the compatibility condition, given by (5) (for uniformly distributed loads).

By substituting (26) into (13), the longitudinal displacement is given by

$$u = u_C + \frac{1}{G}\left(\int_0^y J^*\,\mathrm{d}y - \frac{1}{A}\int_0^y A^*\,\mathrm{d}y\right)\frac{q_x}{t}, \tag{29}$$

where

$$\int_0^y A^*\,\mathrm{d}y = \begin{cases} \dfrac{ty}{2}(2b - y) & (0 \le y \le b), \\ -\dfrac{ty}{2}(2b + y) & (0 \ge y \ge -b); \end{cases} \tag{30}$$

$$\int_0^y J^*\,\mathrm{d}y = \begin{cases} \frac{1}{2}y \mid -\frac{1}{2}(y - c) & (0 \le y \le b), \\ -\frac{1}{2}y \mid +\frac{1}{2}(y + c) & (0 \ge y \ge -b). \end{cases} \tag{31}$$

The transverse displacement, after substitution of (21), (24) and (26) into (19), reads

$$v = -\nu_y \frac{N}{AE_y} y. \tag{32}$$

4. Linearly distributed loads

If the shear stress is given by (26), but $q_x \neq \mathrm{const.}$, then from (16)

$$\sigma_y = \sigma_{y0} - \frac{1}{t}\left(\int_0^y J^*\,\mathrm{d}y - \frac{1}{A}\int_0^y A^*\,\mathrm{d}y\right)\frac{\mathrm{d}q_x}{\mathrm{d}x}, \tag{33}$$

where

$$\sigma_{y0} = \frac{1}{t}\left(\int_b J^*\,\mathrm{d}y - \frac{1}{A}\int_b A^*\mathrm{d}y\right)\frac{\mathrm{d}q_x}{\mathrm{d}x} = \frac{b}{4t}(2\varphi - 1)\frac{\mathrm{d}q_x}{\mathrm{d}x}, \tag{34}$$

where $\varphi = (c/b)$. From (17) one obtains

$$\sigma_x = E_x\frac{\mathrm{d}u_C}{\mathrm{d}x} + \nu_x\sigma_{y0} + \frac{1}{t}\left(\frac{E_x}{G} - \nu_x\right)\left(\int_0^y J^*\,\mathrm{d}y - \frac{1}{A}\int_0^y A^*\,\mathrm{d}y\right)\frac{\mathrm{d}q_x}{\mathrm{d}x} \tag{35}$$

and from (11)

$$E_x A\frac{\mathrm{d}u_C}{\mathrm{d}x} - \frac{E_x}{G}A\kappa\frac{\mathrm{d}q_x}{\mathrm{d}x} = N, \tag{36}$$

where

$$\kappa = \frac{G}{E_x}\left(\frac{E_x}{G} - \nu_x\right)K - \nu_x\frac{Gb}{4E_x t}(2\varphi - 1), \tag{37}$$

where

$$K = \frac{1}{At}\left[\frac{1}{A}\int_A\left(\int_0^y A^*\,\mathrm{d}y\right)\mathrm{d}A - \int_A\left(\int_0^y J^*\,\mathrm{d}y\right)\mathrm{d}A\right] = -\frac{b}{12t}[1 - 3(1 - \varphi)^2]. \tag{38}$$

Finally, one may write

$$\sigma_x = \frac{N}{A} + \left(\frac{E_x}{G} - \nu_x\right)K\frac{\mathrm{d}q_x}{\mathrm{d}x} + \frac{1}{t}\left(\frac{E_x}{G} - \nu_x\right)\left(\int_0^y J^*\,\mathrm{d}y - \frac{1}{A}\int_0^y A^*\,\mathrm{d}y\right)\frac{\mathrm{d}q_x}{\mathrm{d}x}. \tag{39}$$

The stresses given by (26), for $q_x \neq$ const., (33), (34) and (39) satisfy the compatibility condition, given by (5). The equilibrium condition, given by (9), will be satisfied if

$$\frac{\mathrm{d}q_x}{\mathrm{d}x} = \text{const.} \tag{40}$$

Then, taking into account (8), the line loads T must be distributed linearly. Differentiating (36) one obtains (25), where q_x is now a linear function.

The longitudinal displacement is given by (29), where q_x is a linear function.

The displacement u_C can be expressed as

$$u_C = u_p + u_a, \tag{41}$$

where

$$\frac{\mathrm{d}u_p}{\mathrm{d}x} = \frac{N}{EA}, \quad \frac{\mathrm{d}^2u_p}{\mathrm{d}x^2} = -\frac{q_x}{EA} \tag{42}$$

and

$$\frac{\mathrm{d}u_a}{\mathrm{d}x} = \frac{\kappa}{G}\frac{\mathrm{d}q_x}{\mathrm{d}x}. \tag{43}$$

The displacement u_p corresponds formally to the displacement u_C when q_x = const. The displacement u_a is an additional displacement, which can be defined as

$$u_a = \frac{\kappa}{G}q_x + C, \tag{44}$$

where C is a constant of integration, or a displacement of the strip as a rigid body. The displacement given by (29) may then be rewritten as follows

$$u = u_p + \frac{\kappa}{G}q_x + C + \frac{1}{G}\left(\int_0^y J^*\,\mathrm{d}y - \frac{1}{A}\int_0^y A^*\,\mathrm{d}y\right)\frac{q_x}{t}. \tag{45}$$

The transverse displacement, after substitution of (26), (34) and (36) into (19), reads

$$v = \frac{1}{E_y}\left\{-\nu_y\frac{N}{A}y - \lambda y\frac{\mathrm{d}q_x}{\mathrm{d}x} - \frac{1}{t}\left[1 + \nu_y\left(\frac{E_x}{G} - \nu_x\right)\right]\left[\int_0^y\int_0^y J^*\,\mathrm{d}y - \frac{1}{A}\int_0^y\int_0^y A^*\,\mathrm{d}x\right]\right\}\frac{\mathrm{d}q_x}{\mathrm{d}x}, \tag{46}$$

where

$$\lambda = \nu_y\left(\frac{E_x}{G} - \nu_x\right)K - \frac{b}{4t}(2\varphi - 1) \tag{47}$$

and

$$\int_0^y \int_0^y J^* \, \mathrm{d}y \, \mathrm{d}y = \begin{cases} \frac{1}{4}y^2 \mid -\frac{1}{4}(y-c)^2 & (0 \le y \le b), \\ -\frac{1}{4}y^2 \mid +\frac{1}{4}(y+c)^2 & (0 \ge y \ge -b); \end{cases} \tag{48}$$

$$\int_0^y \int_0^y A^* \, \mathrm{d}y \, \mathrm{d}y = \begin{cases} \dfrac{ty^2}{6}(3b - y) & (0 \le y \le b), \\ -\dfrac{ty^2}{6}(3b + y) & (0 \ge y \ge -b). \end{cases} \tag{49}$$

The stresses, finally can be written as follow ($0 \le \eta \le b$)

$$\tau_{xy} = [\eta \mid -(\eta - \varphi)^0] \frac{q_x}{2t}, \tag{50}$$

$$\sigma_x = \frac{N}{A} - \frac{b}{12lt}\left(\frac{E_x}{G} - \nu_x\right)[1 - 3(1-\varphi)^2 - 3\eta^2 \mid +6(\eta - \varphi)] \frac{\mathrm{d}q_x}{\mathrm{d}\xi}, \tag{51}$$

$$\sigma_y = \frac{b}{4lt}[2\varphi - 1 - \eta^2 \mid +2(\eta - \varphi)] \frac{\mathrm{d}q_x}{\mathrm{d}\xi}, \tag{52}$$

where $\xi = (x/l)$ and $\eta = (y/b)$. The displacements may be written as ($0 \le \eta \le b$)

$$u = u_p + C - \frac{b}{12E_y t}\left\{\left(\frac{E_x}{G} - \nu_x\right)[1 - 3(1-\varphi)^2] - 3\nu_x(1 - 2\varphi) - 3\frac{E_x}{G}[\eta^2 \mid -2(\eta - \varphi)]\right\} q_x, \tag{53}$$

$$\begin{aligned} v = {} & -\nu_y \frac{N}{2E_y t}\eta + \frac{b^2}{12E_y lt}\left\{\left[3(2\varphi - 1) - \nu_y\left(\frac{E_x}{G} - \nu_x\right)(3\varphi^2 - 6\varphi + 2)\right]\eta \right. \\ & \left. - \left[1 + \nu_y\left(\frac{E_x}{G} - \nu_x\right)\right][\eta^3 \mid -3(\eta - \varphi)^2]\right\} \frac{\mathrm{d}q_x}{\mathrm{d}\xi}. \end{aligned} \tag{54}$$

For $\varphi = 1$, one obtains the stresses and displacements for the strip loaded along its longitudinal edges ($0 \le \eta \le b$):

$$\tau_{xy} = \eta \frac{q_x}{2t}, \tag{55}$$

$$\sigma_x = \frac{N}{A} - \frac{b}{12lt}\left(\frac{E_x}{G} - \nu_x\right)(1 - 3\eta^2) \frac{\mathrm{d}q_x}{\mathrm{d}\xi}, \tag{56}$$

$$\sigma_y = \frac{b}{4lt}(1 - \eta^2) \frac{\mathrm{d}q_x}{\mathrm{d}\xi}, \tag{57}$$

$$u = u_p + C - \frac{b}{12E_x t}\left(\frac{E_x}{G} + 2\nu_x - 3\frac{E_x}{G}\eta^2\right)q_x, \tag{58}$$

$$v = -\nu_y \frac{N}{2E_y t}\eta + \frac{b^2}{12E_y lt}\left\{\left[3 + \nu_y\left(\frac{E_x}{G} - \nu_x\right)\right]\eta - \left[1 + \nu_y\left(\frac{E_x}{G} - \nu_x\right)\right]\eta^3\right\}\frac{dq_x}{d\xi}. \tag{59}$$

For isotropic materials, when

$$E_x = E_y = E, \quad \nu_x = \nu_y = \nu, \quad \frac{E}{G} = 2(1+\nu), \tag{60}$$

the longitudinal normal stress, given by (56), takes the form

$$\sigma_x = \frac{N}{A} - (2+\nu)\frac{b}{12lt}[1 - 3(1-\eta)^2]\frac{dq_x}{d\xi}, \tag{61}$$

where the stress depends on materials. On the other hand, it is well known from the plane theory of elasticity that in this case the stresses should be independent of materials. Hence, the stress (56) will be corrected, as follows

$$\sigma_x = \frac{N}{A} - \frac{b}{12lt}\left(\frac{E_x}{G} - 2\nu_x\right)(1 - 3\eta^2)\frac{dq_x}{d\xi}. \tag{62}$$

Taking into account (60), for isotropic materials then one has

$$\sigma_x = \frac{N}{A} - \frac{b}{6lt}(1 - 3\eta^2)\frac{dq_x}{d\xi}. \tag{63}$$

The shear stress and the transverse normal stress are obtained independent of materials already, given by (55) and (57). The displacements may be taken as (58) and (59). For isotropic materials, taking into account (60), then one has

$$u = u_p + C - \frac{b}{6Et}[1 + 2\nu - 3(1+\nu)\eta^2]q_x, \tag{64}$$

$$v = -\nu\frac{N}{2Et}\eta + \frac{b^2}{12Elt}\{[3 + \nu(2+\nu)]\eta - [1 + \nu(2+\nu)]\eta^3\}\frac{dq_x}{d\xi}. \tag{65}$$

Now, the general solution for longitudinal normal stress given by (51) should be corrected, since the stress (62) should be a special case of such general solution. Therefore, a function $\Psi(\eta)$ in (51) will be introduced:

$$\sigma_x = \frac{N}{A} - \frac{b}{12lt}\left(\frac{E_x}{G} - \nu_x\right)[1 - 3(1-\varphi)^2 - 3\eta^2 + \Psi \mid +6(\eta - \varphi)]\frac{dq_x}{d\xi}. \tag{66}$$

Then, the function Ψ may be obtained by equating (66), for $\varphi = 1$, and (62):

$$\Psi = -(1 - 3\eta^2)\frac{\nu_x}{\dfrac{E_x}{G} - \nu_x}. \tag{67}$$

For isotropic materials one has

$$\sigma_x = \frac{N}{A} - \frac{b}{12lt}(2+\nu)[1 - 3(1-\varphi)^2 - 3\eta^2 + \Psi \mid +6(\eta - \varphi)]\frac{\mathrm{d}q_x}{\mathrm{d}\xi}, \tag{68}$$

where

$$\Psi = -(1-3\eta^2)\frac{\nu}{2+\nu}. \tag{69}$$

For $\varphi = 0$, the case of loading along the central longitudinal section, the stress (66), taking into account (67), becomes

$$\sigma_x = \frac{N}{A} - \frac{b}{12lt}\left[\left(\frac{E_x}{G} - 2\nu_x\right)(1-3\eta^2) - 3\left(\frac{E_x}{G} - \nu_x\right)(1-2\eta)\right]\frac{\mathrm{d}q_x}{\mathrm{d}\xi}. \tag{70}$$

The other components may be obtained from (50), (52)–(54):

$$\tau_{xy} = -(1-\eta)\frac{q_x}{2t}, \tag{71}$$

$$\sigma_y = -\frac{b}{4lt}(1-\eta)^2\frac{\mathrm{d}q_x}{\mathrm{d}\xi}, \tag{72}$$

$$u = u_p + C - \frac{b}{12E_x t}\left[\frac{E_x}{G} - \nu_x - 3\frac{E_x}{G}(1-\eta)^2\right]q_x, \tag{73}$$

$$v = -\nu_y\frac{N}{E_y t}\eta - \frac{b^2}{12E_y lt}\left\{\left[3 + 2\nu_y\left(\frac{E_x}{G} - \nu_x\right)\right]\eta + \left[1 + \nu_y\left(\frac{E_y}{G} - \nu_x\right)\right](3-\eta)\eta^2\right\}\frac{\mathrm{d}q_x}{\mathrm{d}\xi} \tag{74}$$

where $q_x = T$. For isotropic materials one has

$$\sigma_x = \frac{N}{A} - \frac{b}{12lt}(2+\nu)\left[(1-3\eta^2)\frac{2}{2+\nu} - 3(1-2\eta)\right]\frac{\mathrm{d}q_x}{\mathrm{d}\xi}, \tag{75}$$

$$\tau_{xy} = -(1-\eta)\frac{q_x}{2t}, \tag{76}$$

$$\sigma_y = -\frac{b}{4lt}(1-\eta)^2\frac{\mathrm{d}q_x}{\mathrm{d}\xi}, \tag{77}$$

$$u = u_p + C - \frac{b}{12Et}[2 + \nu - 6(1+\nu)(1-\eta^2)]q_x, \tag{78}$$

$$v = -\nu\frac{N}{Et}\eta - \frac{b^2}{12Elt}\{[3 + 2\nu(2+\nu)]\eta + [1 + \nu(2+\nu)](3-\eta)\eta^2\}\frac{\mathrm{d}q_x}{\mathrm{d}\xi}. \tag{79}$$

The corrected longitudinal normal stresses satisfy the equilibrium conditions (9) and (12).

5. Boundary conditions

Unknown values N and u_p are average values, and can be obtained from boundary conditions, as follows

$$u_p = \tilde{u}_p, \quad N = \tilde{N}, \tag{80}$$

where $\tilde{u}_p$ and $\tilde{N}$ are given values, at a cross-section.

Uniformly distributed loads

- $u_p = 0 \quad (N \neq 0)$. (81)

Then from obtained expressions for stresses and displacements:

$$\sigma_y = 0, \quad \tau_{xy} \neq 0, \quad \sigma_x \neq 0, \quad u \neq 0, \quad v \neq 0. \tag{82}$$

- $N = 0 \quad (u_p \neq 0)$: (83)

$$\sigma_x = 0, \quad \sigma_y = 0, \quad v = 0, \quad \tau_{xy} \neq 0, \quad u \neq 0. \tag{84}$$

Linearly distributed loads—for $q_x = 0$

- $u_p = 0 \quad (N \neq 0)$: (85)

$$\tau_{xy} = 0, \quad u = 0; \quad \sigma_x \neq 0, \quad \sigma_y \neq 0, \quad v \neq 0. \tag{86}$$

- $N = 0 \quad (u_p \neq 0)$: (87)

$$\tau_{xy} = 0, \quad \sigma_x^* \neq 0, \quad \sigma_y \neq 0, \quad u \neq 0, \quad v \neq 0. \tag{88}$$

Linearly distributed loads—for $q_x \neq 0$

- $u_p = 0 \quad (N \neq 0)$: (89)

$$\tau_{xy} \neq 0, \quad \sigma_x \neq 0, \quad \sigma_y \neq 0, \quad u \neq 0, \quad v \neq 0. \tag{90}$$

- $N = 0 \quad (u_p \neq 0)$: (91)

$$\tau_{xy} \neq 0, \quad \sigma_x^* \neq 0, \quad \sigma_y \neq 0, \quad u \neq 0, \quad v \neq 0. \tag{92}$$

Here (*) denotes that the main vector of σ_x is equal to zero. The condition given by (83) and (84) corresponds to the 'transverse displacement restrained support' (in the plane of antisymmetry of an antisymmetrically loaded multispan girder—Shade, 1951). The condition given by (85) and (86) corresponds to the 'longitudinal displacement restrained support' (in the plane of symmetry of a symmetrically loaded multispan girder, inside a span—von Karman, 1924). The condition given by (91) and (92) corresponds to the condition (83) and (84) in an average way; the influence of self-equilibrated longitudinal forces on the rest of the strip (the main vector of the longitudinal normal stress σ_x is equal to zero) should be discussed in accordance with the Saint Venant's principle. The conditions (81) and

(82) and (89) and (90) can be compared, in an average way, to the condition (85) and (86). The condition given by (87) and (88) corresponds in an average way to the so-called free–free end—free of the longitudinal normal stress, free of the shear stress (as it is well known, this condition does not admit trigonometric stress-functions in the exact solution—Shade, 1951; Bhattacharyya and Vendham, 1987).

6. Examples

6.1. Example 1

A strip loaded by linearly distributed line loads $T = T(1)\xi$ along its longitudinal edges: (a) $u_p = 0$ at $\xi = 0$ and $N = 0$ at $\xi = 1$; (b) $u_p = 0$ at $\xi = 0$ and at $\xi = 1$, respectively.

(a) The boundary conditions are given by (85) and (86) for $\xi = 0$ and by (91) and (92) for $\xi = 1$.
Determination of the function N and u_p is a statically determinate problem:

$$N = \frac{q_x(1)l}{2}(1 - \xi^2), \quad u_p = \frac{q_x(1)l^2}{6E_xA}\xi(3 - \xi^2), \quad q_x = q_x(1)\xi, \quad q_x(1) = 2T(1); \tag{93}$$

for isotropic materials $E_x = E$. The stresses and displacements are given by (55), (57)–(62); for isotropic materials by (55), (57) and (63)–(65).
The extreme longitudinal normal stresses, at $\eta = 0$ and $\eta = 1$, respectively, can be expressed as follows

$$\sigma_x(0) = \frac{N}{A}\psi(0), \quad \sigma_x(1) = \frac{N}{A}\psi(1), \tag{94}$$

where

$$\psi(0) = 1 - \frac{1}{3}\left(\frac{E_x}{G} - 2\nu_x\right)\left(\frac{b}{l}\right)^2 \frac{1}{1 - \xi^2}, \quad \psi(1) = 1 + \frac{2}{3}\left(\frac{E_x}{G} - 2\nu_x\right)\left(\frac{b}{l}\right)^2 \frac{1}{1 - \xi^2}. \tag{95}$$

For isotropic materials:

$$\psi(0) = 1 - \frac{2}{3}\left(\frac{b}{l}\right)^2 \frac{1}{1 - \xi^2}, \quad \psi(1) = 1 + \frac{4}{3}\left(\frac{b}{l}\right)^2 \frac{1}{1 - \xi^2}. \tag{96}$$

The solution corresponds, in an average way, to the exact solution for the flanges of antisymmetrically vertically loaded multispan thin-walled girders, with close (Fig. 3a) or open cross-sections (Fig. 3b). For $\xi = 0$, the solution will agree with the exact solution, for l/b not too small (Saint Venant's principle). The solution may also be used in the analysis of isolated simple supported thin-walled girders—where boundary conditions (at supports) are given (by the plane theory of elasticity) also in an average way (Filin, 1978).

(b) The boundary conditions are given by (85) and (86) for $\xi = 0$ and by (89) and (90) for $\xi = 1$; determination of the functions N and u_p is a statically indeterminate problem:

$$N = \frac{q_x(1)l}{6}(1 - 3\xi^2), \quad u_p = \frac{q_x(1)l^2}{6E_xA}\xi(1 - \xi^2). \tag{97}$$

In this case:

$$\psi(0) = 1 - \left(\frac{E_x}{G} - 2\nu_x\right)\left(\frac{b}{l}\right)^2 \frac{1}{1-3\xi^2}, \quad \psi(1) = 1 + 2\left(\frac{E_x}{G} - 2\nu_x\right)\left(\frac{b}{l}\right)^2 \frac{1}{1-3\xi^2}. \tag{98}$$

For isotropic materials:

$$\psi(0) = 1 - 2\left(\frac{b}{l}\right)^2 \frac{1}{1-3\xi^2}, \quad \psi(1) = 1 + 4\left(\frac{b}{l}\right)^2 \frac{1}{1-3\xi^2}. \tag{99}$$

The solution corresponds, in an average way, to the exact solution for the flanges of symmetrically vertically loaded multispan thin-walled girders (symmetrically with respect to a midspan) with close (Fig. 3a) or open cross-sections (Fig. 3b). For $\xi = 0$, the solution will agree with the exact solution, for l/b not too small. The solution may also be used in the analysis of isolated girders with clamped ends—where boundary conditions (at supports) are given (by the plan theory of elasticity) also in an average way (Filin, 1978).

6.2. Example 2

A strip loaded by linearly distributed line loads $T = T(1)\xi$ along its central longitudinal section; $u_p = 0$ at $\xi = 0$ and $N = 0$ at $\xi = 1$.

The load and boundary conditions are as in Example 1(a): N, u_p and q_x given by (93), where $q_x(1) = T(1)$.

The stresses and displacements are given by (70)–(74); for isotropic materials by (75)–(79). The extreme longitudinal normal stresses can be obtained by (94), where

$$\psi(0) = 1 + \frac{2}{3}\left(\frac{E_x}{G} - \frac{\nu_x}{2}\right)\left(\frac{b}{l}\right)^2 \frac{1}{1-\xi^2}, \quad \psi(1) = 1 - \frac{1}{3}\left(\frac{E_x}{G} - \frac{\nu_x}{2}\right)\left(\frac{b}{l}\right)^2 \frac{1}{1-\xi^2}; \tag{100}$$

for isotropic materials:

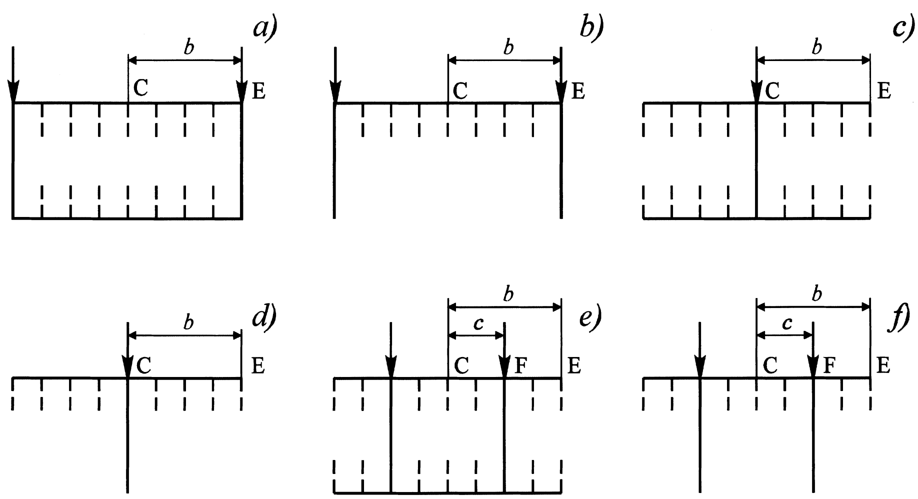

Fig. 3. Cross-sections of thin-walled girders.

$$\psi(0) = 1 + \frac{2}{3}\left(2 + \frac{3}{2}\nu\right)\left(\frac{b}{l}\right)^2 \frac{1}{1 - \xi^2}, \quad \psi(1) = 1 - \frac{1}{3}\left(2 + \frac{3}{2}\nu\right)\left(\frac{b}{l}\right)^2 \frac{1}{1 - \xi^2}. \tag{101}$$

The extreme transverse normal stress can be expressed as follows

$$\sigma_y(0) = -\frac{N}{A}\chi(0), \tag{102}$$

where

$$\chi(0) = \left(\frac{b}{l}\right)^2 \frac{1}{1 - \xi^2}. \tag{103}$$

The solution corresponds, in an average way, to the exact solution for the flanges of antisymmetrically vertically loaded multispan thin-walled girders with open cross-sections (Fig. 3c and d).

6.3. Example 3

A strip loaded by linearly distributed line loads $T = T(1)(\xi)$ along the longitudinal sections ($0 \leq \eta \leq 1$, $-1 \leq \eta \leq 0$): (a) $u_p = 0$ at $\xi = 0$ and $N = 0$ at $\xi = 1$; (b) $u_p = 0$ at $\xi = 0$ and at $\xi = 1$, respectively.

(b) The loads and boundary conditions are given as in Example 1(a); N, u_p and q_x are given by (93).

The stresses and displacements are given by (50), (52)–(54), (66) and (67); for isotropic materials by (50), (52)–(54), (68) and (69). The extreme longitudinal stress, at $\eta = \varphi$, may be obtained by (94), where

$$\psi(\varphi) = 1 + \frac{1}{3}\left[3\left(\frac{E_x}{G} - \nu_x\right)(1 - \varphi)^2 - \left(\frac{E_x}{G} - 2\nu_x\right)(1 - 3\varphi^2)\right]\left(\frac{b}{l}\right)^2 \frac{1}{1 - \xi^2}, \tag{104}$$

and

$$\psi(\varphi) = 1 + \frac{1}{3}[3(2 + \nu)(1 - \varphi)^2 - 2(1 - 3\varphi^2)]\left(\frac{b}{l}\right)^2 \frac{1}{1 - \xi^2}. \tag{105}$$

The solution corresponds, in an average way, to the exact solution for the flanges of antisymmetrically vertically loaded multispan thin-walled girders, with closed open cross-sections (Fig. 3e) or open cross-sections (Fig. 3f).

(b) The loads and boundary conditions as in Example 1(b); N, u_p and q_x given by (97).

In this case:

$$\psi(\varphi) = 1 + \left[3\left(\frac{E_x}{G} - \nu_x\right)(1 - \varphi)^2 - \left(\frac{E_x}{G} - 2\nu_x\right)(1 - 3\varphi^2)\right]\left(\frac{b}{l}\right)^2 \frac{1}{1 - 3\xi^2}, \tag{106}$$

and

$$\psi(\varphi) = 1 + [3(2 + \nu)(1 - \varphi)^2 - 2(1 - 3\varphi^2)]\left(\frac{b}{l}\right)^2 \frac{1}{1 - 3\xi^2}. \tag{107}$$

The solution corresponds, in an average way, to the exact solution for the flanges of symmetrically vertically loaded multispan thin-walled girders, with closed open cross-sections (Fig. 3e) or open cross-sections (Fig. 3f). For $\xi = 0$, the solution will agree with the exact solution, for l/b not too small. The solution can also be used in the analysis of isolated girders with clamped ends, where boundary conditions (at supports) are given (by the plane theory of elasticity) also in an average way (Filin, 1978).

6.4. Example 4

A strip loaded by linearly distributed line loads $T = T(0)(1-\xi)$ along the central longitudinal section; $u_p = 0$ at $\xi = 0$ and $N = 0$ at $\xi = 1$.

The boundary conditions are given by (89) and (90) for $\xi = 0$ and by (87) and (88) for $\xi = 1$. Determination of the functions N and u_p is a statically determinate problem:

$$N = \frac{q_x(0)l}{2}(1-\xi)^2, \quad u_p = \frac{q_x(0)l^2}{6E_xA}\xi[3-\xi(3-\xi)],$$

$$q_x = q_x(0)(1-\xi), \quad q_x(0) = T(0). \tag{108}$$

In this case:

$$\psi(0) = 1 + \frac{2}{3}\left(\frac{E_x}{G} - \frac{\nu_x}{2}\right)\left(\frac{b}{l}\right)^2 \frac{1}{(1-\xi)^2}, \quad \psi(1) = 1 - \frac{1}{3}\left(\frac{E_x}{G} - \frac{\nu_x}{2}\right)\left(\frac{b}{l}\right)^2 \frac{1}{(1-\xi)^2}, \tag{109}$$

and

$$\psi(0) = 1 + \frac{2}{3}\left(2 + \frac{3}{2}\nu\right)\left(\frac{b}{l}\right)^2 \frac{1}{(1-\xi)^2}, \quad \psi(1) = 1 - \frac{1}{3}\left(2 + \frac{3}{2}\nu\right)\left(\frac{b}{l}\right)^2 \frac{1}{(1-\xi)^2}. \tag{110}$$

The solution can be used in the analysis of the flanges of consoles with open cross-sections (Fig. 3c and d), where the boundary condition at $\xi = 0$ is given (by the plane theory of elasticity) also in an average way (Filin, 1978).

7. Comparison of results

The results obtained for Example 1 are compared to the results of the plane theory of elasticity. The loads are presented by Fourier series by cosine mode shapes for Case (a), where $\sigma_x = 0$, $\sigma_y = 0$ and $v = 0$, $\tau_{xy} \neq 0$, $u \neq 0$, at $\xi = \pm 1$, and by sine mode shapes for Case (b), where $\tau_{xy} = 0$ and $u = 0$, $\sigma_x \neq 0$, $\sigma_y \neq 0$, $v \neq 0$, at $\xi = \pm 1$.

For Case (a) the following ratios $\psi(1)$, at $\xi = 0$, according to (94), can be obtained (Kurdyumov et al., 1963):

$$\psi(1)] = \frac{\pi}{2} \frac{\sum_{n}^{1,\,3\ldots} \frac{1}{n^2}\vartheta_n^* \sin n\frac{\pi}{2}}{\sum_{n}^{1,\,3\ldots} \frac{1}{n^3} \sin n\frac{\pi}{2}} \tag{111}$$

for orthotropic materials;

$$\psi(1)=\frac{\pi b}{l}\times\frac{\sum_{n}^{1,\,3\ldots}\frac{1}{n^2}\vartheta_n\ \sin\ n\frac{\pi}{2}}{\sum_{n}^{1,\,3\ldots}\frac{1}{n^3}\sin\ n\frac{\pi}{2}} \tag{112}$$

for isotropic materials.

For Case (b) it may be obtained:

$$\psi(1)=\frac{\pi}{2}\times\frac{\sum_{n}^{2,\,4\ldots}\frac{1}{n}\vartheta_n^*\ \cos\ n\frac{\pi}{2}}{\sum_{n}^{2,\,4\ldots}\frac{1}{n^2}\cos\ n\frac{\pi}{2}} \tag{113}$$

for orthotropic materials;

$$\psi(1)=\frac{\pi b}{l}\times\frac{\sum_{n}^{2,\,4\ldots}\frac{1}{n}\vartheta_n\ \cos\ \frac{n\pi}{2}}{\sum_{n}^{2,\,4\ldots}\frac{1}{n^2}\cos\ \frac{n\pi}{2}} \tag{114}$$

for isotropic materials. Here

$$\vartheta_n^*=\frac{(g_1^2-g_2^2)\text{ch}\ u_n\ \text{ch}\ v_n}{g_1\ \text{sh}\ u_n\ \text{ch}\ v_n-g_2\ \text{ch}\ u_n\ \text{sh}\ v_n},\quad \vartheta_n=\frac{1+\text{ch}\ \alpha_n}{\alpha_n+\text{sh}\ \alpha_n}, \tag{115}$$

where

$$u_n=\frac{g_1 n\pi}{2},\quad v_n=\frac{g_2 n\pi}{2},\quad \alpha_n=\frac{n\pi b}{l}$$

$$g_1=\sqrt{\frac{\Omega}{\Gamma}}\times\sqrt{1+\sqrt{1-\frac{1}{\Omega^2}}},\quad g_2=\sqrt{\frac{\Omega}{\Gamma}}\times\sqrt{1-\sqrt{1-\frac{1}{\Omega^2}}},\quad \frac{1}{\Omega^2}<1, \tag{116}$$

where

$$\Gamma=\left(\frac{l}{b}\right)^2\sqrt{\frac{\delta_1}{\delta_2}},\quad \Omega=\frac{\delta_3}{\sqrt{\delta_1\delta_2}},\quad \delta_1=\frac{1}{E_x},\quad \delta_2=\frac{1}{E_y},\quad \delta_3=\frac{1}{2G}-\frac{\nu_x}{E_x}. \tag{117}$$

The results of comparison for Case (a), for the orthotropic materials $E_x=18$ GPa, $E_y=17$ GPa, $G=1.9$ GPa, $\nu_x=0.118$, and $E_x=315$ GPa, $E_y=210$ GPa, $G=80.77$ GPa, $\nu_x=0.367$, are presented in Tables 1 and 2, respectively. For Case (b) the results are presented in Tables 3 and 4, respectively. The results of comparison for isotropic materials for Case (a) are presented in Table 5, and for Case (b) in Table 6, respectively.

The results obtained for Example 2, for the isotropic material $E=210$ GPa, $\nu=0.3$, are compared to the results of a finite element analysis (FEA). In proceeding with FEA rectangular meshes containing 4050 elements (quadrilateral plane stress) and 4186 nodes, with two degrees-of-freedom (x-, y-

Table 1
Value of $\psi(1)$, at $\xi = 0$, for Example 1(a) for $E_x = 18$, $E_y = 17$, $G = 1.9$ GPa, $\nu_x = 0.118$

l/b	(95)	(111)	n	Δ[a]
2	2.540	2.318	197	8.7
3	1.684	1.650	197	2.0
4	1.385	1.378	197	0.5
5	1.246	1.245	197	0.1
6	1.171	–		
7	1.126	–		
8	1.098	–		
9	1.077	–		
10	1.062	–		

[a] $\Delta = \frac{|(111)-(95)|}{(95)} 100.$

Table 2
Values of $\psi(1)$, at $\xi = 0$, for Example 1(a) for $E_x = 315$, $E_y = 210$, $G = 80.77$ GPa, $\nu_x = 0.367$

l/b	(95)	(111)	n	Δ[a]
2	1.528	1.538	103	0.7
3	1.235	1.235	39	0.0
4	1.132	1.132	31	0.0
5	1.084	1.084	65	0.0
6	1.059	–		
7	1.043	–		
8	1.033	–		
9	1.026	–		
10	1.021	–		

[a] $\Delta = \frac{|(111)-(95)|}{(95)} 100.$

Table 3
Value of $\psi(1)$, at $\xi = 0$, for Example 1(b): $E_x = 18$, $E_y = 17$, $G = 1.9$ GPa, $\nu_x = 0.118$

l/b	(98)	(113)	n	Δ[a]
2	5.619	4.150/4.102	266/268	27
3	3.053	2.714/2.735	396/398	11
4	2.155	2.064/2.076	536/538	3.9
5	1.739	1.711/1.720	600/602	1.4
6	1.513	1.503/1.510	600/602	0.4
7	1.378	1.372/1.378	600/602	0.2
8	1.289	–		
9	1.228	–		
10	1.185	–		

[a] $\Delta = \frac{|(113)-(98)|}{(98)} 100.$

Table 4
Values of $\psi(1)$, at $\xi = 0$, for Example 1(b): $E_x = 315$, $E_y = 210$, $G = 80.77$ GPa, $\nu_x = 0.367$

l/b	(98)	(113)	n	Δ[a]
2	2.583	2.643/2.677	264/266	3.0
3	1.704	1.727/1.712	402/404	0.9
4	1.396	1.401/1.393	536/538	0.1
5	1.253	1.256/1.250	598/600	0.0
6	1.176	1.178/1.173	598/600	0.0
7	1.129	1.131/1.127	598/600	0.0
8	1.098	–		
9	1.078	–		
10	1.063	–		

[a] $\Delta = \frac{|(113)-(98)|}{(98)} 100.$

Table 5
Values of $\psi(1)$, at $\xi = 0$, for Example 1(a) for isotropic materials

l/b	(96)	(112)	n	Δ[a]
2	1.333	1.344	37	0.8
3	1.148	1.148	35	0.0
4	1.083	1.083	29	0.0
5	1.053	1.053	33	0.0
6	1.037	–		
7	1.027	–		
8	1.021	–		
9	1.017	–		
10	1.013	–		

[a] $\Delta = \frac{|(112)-(96)|}{(96)} 100.$

Table 6
Values of $\psi(1)$, at $\xi = 0$, for Example 1(b) for isotropic materials

l/b	(99)	(114)	n	Δ[a]
2	2.000	2.105/2.122	448/450	5.7
3	1.444	1.432/1.444	448/450	0.5
4	1.250	1.234/1.242	448/450	1.0
5	1.160	1.147/1.154	448/450	0.8
6	1.111	–		
7	1.082	–		
8	1.063	–		
9	1.050	–		
10	1.040	–		

[a] $\Delta = \frac{|(114)-(99)|}{(99)} 100.$

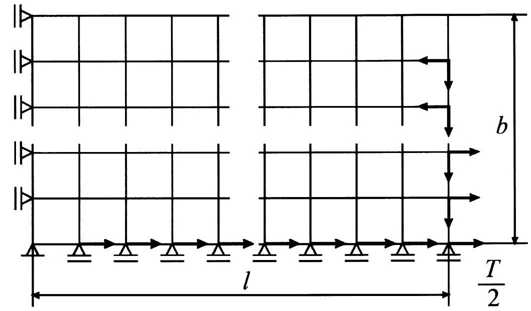

Fig. 4. Finite element model for Example 2.

displacements) per node, for $l/b = 2$, and 2700 elements and 2821 nodes, for $l/b = 3$, are used (Fig. 4). The sizes of meshes correspond to one quarter of the strips. The model is also loaded along the end at $\xi = 1$, by line loads in accordance with the analytical solutions for the shear stresses, given by (76), and longitudinal normal stresses, given by (75). The results of comparison are shown in Table 7.

For Example 3 the meshes contained (Fig. 5): 8100 elements and 8281 nodes for $l/b = 1$ and 5400 elements and 5551 nodes for $l/b = 3/2$; for $l/b = 2$ and $l/b = 3$ the meshes as for Example 2. The

Table 7
Distributions of $\psi(0)$ and $\chi(0)$, respectively, for Example 2: $E = 210$ GPa, $\nu = 0.3$

x/l	$l/b = 2$					$l/b = 3$				
	$\psi(0)$			$\chi(0)$		$\psi(0)$			$\chi(0)$	
	(101)	FEA	Δ[a]	(103)	FEA	(101)	FEA	Δ[a]	(103)	FEA
0.0	1.408	1.408	0.0	0.250	0.247	1.182	1.182	0.0	0.111	0.110
0.2	1.425	1.425	0.0	0.260	0.256	1.189	1.189	0.0	0.116	0.115
0.4	1.486	1.483	0.2	0.298	0.291	1.216	1.217	0.1	0.132	0.131
0.6	1.638	1.628	0.6	0.391	0.381	1.284	1.282	0.2	0.174	0.169
0.8	2.134	2.105	1.4	0.694	0.686	1.504	1.497	0.5	0.309	0.302

[a] $\Delta = \frac{|\mathrm{FEA}-(101)|}{(101)}100.$

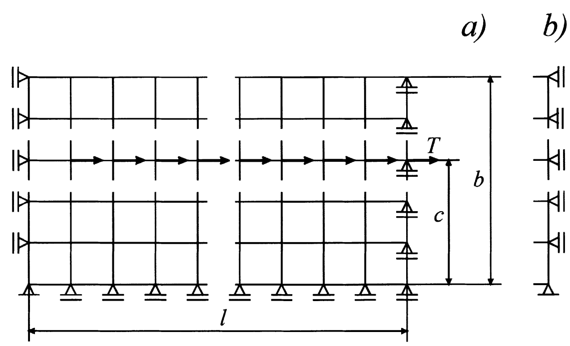

Fig. 5. Finite element model for Example 3.

Table 8
Values of $\psi(2/3)$, at $\xi=0$, for Example 3(a): $E = 210$ GPa, $\nu=0.3$

l/b	(105)	FEA	Δ[a]
1	1.483	1.458	1.7
3/2	1.215	1.213	0.2
2	1.121	1.118	0.3
3	1.054	1.053	0.1

[a] $\Delta = \frac{|\mathrm{FEA}-(105)|}{(105)} 100.$

Table 9
Values of $\psi(2/3)$, at $\xi=0$, for Example 3(b): $E = 210$ GPa, $\nu=0.3$

l/b	(107)	FEA	Δ[a]
1	2.449	2.181	11
3/2	1.644	1.598	2.8
2	1.362	1.349	1.0
3	1.161	1.151	0.9

[a] $\Delta = \frac{|\mathrm{FEA}-(107)|}{(107)} 100.$

boundary conditions for Case (a) are shown in Fig. 5a, and for Case (b) in Fig. 5b. The result for Case (a) are shown in Table 8, and for Case (b) in Table 9.

8. Concluding remarks

A novel approximate analytical method has been applied to estimate stresses and displacements in thin rectangular orthotropic or isotropic strips subjected to tension by linearly distributed line loads. The solution for isotropic strips is given as a special case of the solution for orthotropic strips. The solution for uniformly distributed load can be obtained as a special case of the solution for linearly distributed loads. The strip can be loaded generally (symmetrically) within the longitudinal edges. Loading along the longitudinal edges and the central longitudinal section are given as special cases of the general loading.

The method is approximate due to the introduced assumptions. The reliability of the assumptions is proved especially by Example 2, where distributions of the extreme longitudinal and transverse normal stresses along the strip length are analyzed in comparison with the results of a finite element analysis. For that case, the identical conditions are provided by loading the finite element model at the end cross-section at $\xi=1$ according to the analytic solution.

Detail comparisons with the exact solutions of the plane theory of elasticity and the finite element analysis show an acceptable agreement of the obtained results for extreme normal stresses for various ratios l/b and c/b, by assuming the ratios l/b are not too small.

By assuming an error $\leq 2\%$ with respect to the results of the plane theory of elasticity, the following constraints for l/b can be drawn, for orthotropic and isotropic materials:

$$\frac{l}{b} \geq 2 \quad \text{for } 2 \leq \frac{E_x}{G} - 2\nu_x \leq 3,$$

$$\frac{l}{b} \geq \frac{1}{6}\left(\frac{E_x}{G} - 2\nu_x - 3\right) + 2 \quad \text{for } 3 \leq \frac{E_x}{G} - 2\nu_x \leq 9 \tag{118}$$

for the strip loaded along its longitudinal edges—Example 1(a), where the end at $\xi = 0$ is under the identical conditions with respect to the solution by the plane theory of elasticity, given by (85) and (86); the end at $\xi = 1$ under 'average' conditions, given by (91) and (92);

$$\frac{l}{b} \geq 3 \quad \text{for } 2 \leq \frac{E_x}{G} - 2\nu_x \leq 3,$$

$$\frac{l}{b} \geq \frac{1}{3}\left(\frac{E_x}{G} - 2\nu_x - 3\right) + 3 \quad \text{for } 3 \leq \frac{E_x}{G} - 2\nu_x \leq 9 \tag{119}$$

for the strip loaded along its longitudinal edges—Example 1(b), where the end at $\xi = 0$ is under the identical conditions with respect to the solution by plane theory of elasticity, given by (85) and (86); the end at $\xi = 1$ under 'average' conditions, given by (89) and (90).

The constraints (118) and (119) are obtained by using two orthotropic materials ($E_x = 18$ GPa, $E_y = 17$ GPa, G = 1.9 GPa, $\nu_x = 0.118$, and $E_x = 315$ GPa, $E_y = 210$ GPa, G = 80.77 GPa, $\nu_x = 0.367$), and a linear interpolation, included isotropic materials, as shown in Fig. 6. The constraints (118) are shown in Fig. 6a, the constraints (119) in Fig. 6b.

By assuming an error $\leq 2\%$ with respect to the results of the finite element analysis, the following constraints for l/b are obtained, for isotropic materials:

$$\frac{l}{b} \geq 2 \tag{120}$$

for the strip loaded along its central longitudinal section—Example 2, where the ends are under the identical conditions with respect to FEA, given by (85) and (86) and (91) and (92), respectively;

$$\frac{l}{b} \geq 1 \tag{121}$$

for the strip loaded along longitudinal sections ($\varphi = 2/3$)—Example 3(a), where the end at $\xi = 0$ is under the identical conditions with respect to FEA, given by (85) and (86); the end at $\xi = 1$ under 'average' conditions, given by (91) and (92);

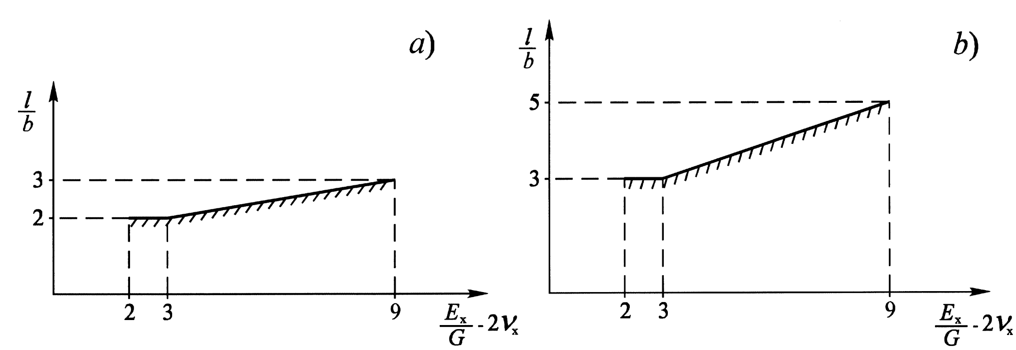

Fig. 6. Constraints to Example 2.

$$\frac{l}{b} \geq 4 \tag{122}$$

for the strip loaded along longitudinal sections ($\varphi = 2/3$)—Example 3(b), where the end at $\xi = 0$ is under the identical conditions with respect to FEA, given by (85) and (86); the end at $\xi = 1$ under 'average' conditions, given by (89) and (90).

As it could be expected, good agreement is obtained, in general, for Case (a); and in some lesser form for Case (b). For Case (a), namely, the main vector of longitudinal normal stresses at $\xi = 1$ is obtained equal to zero, whereby the plane theory of elasticity, as well as by FEA, the longitudinal normal stresses vanish totally (except in Example 2). For Case (b) the average longitudinal displacement at $\xi = 1$ is resulting to zero, whereby the plane theory of elasticity as well as by FEA, the longitudinal displacements are equal in total to zero. Here, must be noted that for Case (b) the solution at $\xi = 1$ by the plane theory of elasticity suffers from stress singularities. It is assumed that the shear stresses at $\xi = 1$ are equal to zero and, on the other hand, the shear loads at $\xi = 1$ are given unequal to zero. This influence on the end at $\xi = 1$ can be more significant than the influence of the self-equilibrated forces (of the longitudinal normal stress) in Case (a).

It may also be noted that better results, in general, are obtained for isotropic materials. Namely, the introduced assumptions are more realistic in the case of isotropic strips. Therefore, the constraints for orthotropic strips are given in a somewhat more severe form. Here, must be noted that material characteristics in the case of structural orthotropy usually are close to the characteristics of the isotropic materials. (The material $E_x = 315$ GPa, $E_y = 210$ GPa, $G = 80.77$ GPa, $\nu_x = 0.367$, corresponds to a typical structural orthotropic strip.)

Finally, the present solutions should be more realistic in the analysis of flanges of isolated beams; especially beams with clamped ends, or consoles. Namely, it is hard to assume that the longitudinal displacement can be totally restrained at such clamped ends. The assumption that only the average longitudinal displacement there is equal to zero seems to be more realistic. The exact solutions of the plane theory of elasticity by polynomials treat this problem in the same way.

In conclusion, the obtained solution for stresses and displacements is simple and analytical. Under given constraints, it can be used in the analysis of flanges of various types of thin-walled beams with relatively small flange ratios l/b, subjected to bending by uniform vertical loads. It should be useful especially in the early design stage of structures when many parts of structures are submitted to optimization processes.

Acknowledgements

This research was supported by the Ministry of Science and Technology of the Republic of Croatia, under Contract No. 120006 (1996).

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